

Quantum discord of 2^n -dimensional Bell-diagonal states

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Abstract

In this study, using the concept of relative entropy as a distance measure of correlations we investigate the important issue of evaluating quantum correlations such as entanglement, dissonance and classical correlations for 2^n -dimensional Bell-diagonal states. We provide an analytical technique, which describes how we find the closest classical states(CCS) and the closest separable states(CSS) for these states. Then analytical results are obtained for quantum discord of 2^n -dimensional Bell-diagonal states. As illustration, some special cases are examined. Finally, we investigate the additivity relation between the different correlations for the separable generalized bloch sphere states.

Keywords: Quantum Discord, Distance Measure of Correlations, Dirac γ matrices, Bipartite Quantum System.

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1 Introduction

Quantum entanglement plays an important role in the quantum communication protocols like teleportation [1, 2], superdense coding [3], remote state preparation [4], cryptography [5] and many more. However, entanglement is not the only correlation that is useful for quantum information processing. Recently, it is found that many tasks, e.g. quantum non-locality without entanglement [6, 7, 8], can be carried out with quantum correlations other than entanglement. It has been shown both theoretically and experimentally [9, 10] that some separable states may speed up certain tasks over their classical counterparts. Recent measures of nonclassical correlations are motivated by different notions of classicality and operational means to quantify nonclassicality. One kind of nonlocal correlation called quantum discord, as introduced by Oliver and Zurek [11, 12], has received much attention recently [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. Most of these works are limited to studies of bipartite correlations only as the concept of discord, which relies on the definition of mutual information, is not defined for multipartite systems. It is well known that the different measures of quantum correlation are not identical and conceptually different. For example, the discord does not coincide with entanglement and a direct comparison of two notions is rather meaningless. Therefore, a unified classification of correlations is in demand. Modi et al.[19], introduced a unified classification of correlations for quantum states which is applicable for multipartite systems. In this unified view, the measure of correlation is based on the idea that the distance from a given state to the closest state without the desired property (e.g. entanglement or discord) is a measure of that property. Finding the CCS is still a very difficult problem and has the same challenged as faced in computing original discord. [26, 27]. The examples of entangled states ρ with analytical expression for the CSSs, discussed in Refs.[28, 29, 30, 31, 32, 33, 34, 35, 36]. The inverse problem to the long standing problem [37] of finding the formula for the CSS was solved in [38] for the case of two qubits and a closed

formula for all entangled states was solved in [39] for all dimensions and for any number of parties. In this paper, we give an efficient procedure so that analytic evaluation of quantum discord of 2^n -dimensional Bell-diagonal states can be performed. Then we find an exact explicit formula for quantum discord of these states. We also show that total correlation for the separable generalized bloch sphere states is subadditive.

This paper is organized as follows. In the next section we introduce the distance measure of correlation and show that the generic state ρ and its CSS or CCS have the same structure. In section 3 the definition of 2^n -dimensional Bell-diagonal states is given and in section 4 we calculate the CCS of 2^n -dimensional Bell-diagonal states and then we find an exact analytical formula for the quantum discord of these states. In the rest of this section to illustrate how the formula can be applied, we give two examples. In section 5 we investigate additivity relations between different correlations for separable generalized bloch sphere states. Concluding remarks and two Appendices close this paper.

2 Distance measure of correlations.

Here, we will follow the approach of [19] to characterize and quantify all kinds of correlations in a quantum state. The definitions of relevant quantities are:

$$\textit{Entanglement} \quad E = \min_{\sigma \in \mathcal{D}} S(\rho \| \sigma), \quad (2.1)$$

$$\textit{Discord} \quad D = \min_{\chi \in \mathcal{C}} S(\rho \| \chi), \quad (2.2)$$

$$\textit{Dissonance} \quad Q = \min_{\chi \in \mathcal{C}} S(\sigma \| \chi), \quad (2.3)$$

$$\textit{Classical correlations} \quad C = \min_{\pi \in \mathcal{P}} S(\chi \| \pi), \quad (2.4)$$

where \mathcal{P} is the set of all product states (i.e., states of the form $\pi = \pi_1 \otimes \pi_2 \otimes \dots \otimes \pi_N$ and π_n is the reduced state of the n th subsystem). \mathcal{C} contains mixtures of locally distinguishable states $\chi = \sum_{k_n} p_{k_1 \dots k_N} |k_1 \dots k_N\rangle \langle k_1 \dots k_N| = \sum_{\vec{k}} p_{\vec{k}} |\vec{k}\rangle \langle \vec{k}|$ where $p_{\vec{k}}$ is a joint probability distribution

and local states $|k_n\rangle$ span an orthonormal basis, \mathcal{D} is the set of all separable states (i.e., states of the form $\sigma = \sum_k p_k \pi_1^k \otimes \pi_2^k \otimes \dots \otimes \pi_N^k$) and $S(x||y) = \text{Tr}(x \log x - x \log y)$ is the relative entropy of x with respect to y .

A bipartite state is called classical if it contains mixtures of locally distinguishable states, and is called separable if it can be represented as a convex combination of product states. Finding out the CSS is a non trivial task[27]. While the set of separable states is apparently convex, this is not the case for the set of classical states then determining the CCS is even more complicated. Here we present an analytical procedure that allows us to obtain the CSS and CCS for 2^n -dimensional Bell-diagonal states. The key idea is to find the minimum distance from a given state ρ to the set of all states without the desire property. The following theorem plays a central role in minimizing the mentioned distance.

Theorem: Given a generic state $\rho \in \mathcal{H} \setminus \mathcal{I}$ and $X \in \mathcal{I}$, $\min S(\rho||X)$ is achieved when ρ and X have common eigenbasis. Here, \mathcal{I} is a special subset of the Hilbert space \mathcal{H} .

To show this, suppose

$$\rho = \sum_i^N \lambda_i |\lambda_i\rangle \langle \lambda_i|, \quad X = \sum_j^N \mu_j |\mu_j\rangle \langle \mu_j|,$$

then we have

$$\min S(\rho||X) = \min[\text{Tr} \rho \log \rho - \text{Tr}(\rho \log X)] = \sum \lambda_i \log \lambda_i - \max_{i,j} \sum \lambda_i |\langle \lambda_i | \mu_j \rangle|^2 \log \mu_j \quad (2.5)$$

Suppose $|\langle \lambda_i | \mu_j \rangle|^2 = q_{ij}$, where

$$\sum_i q_{ij} = 1, \quad \sum_j q_{ij} = 1. \quad (2.6)$$

Now the problem of finding the closest state X to ρ is reduced to the problem

$$\begin{cases} \text{maximize} & \sum_{i,j} \lambda_i q_{ij} \log \mu_j = \lambda^T Q \eta. \\ \text{subject to} & \sum_i q_{ij} = 1, \quad \sum_j q_{ij} = 1 \end{cases} \quad (2.7)$$

where $\eta^T = (\log \mu_1, \log \mu_2, \dots, \log \mu_{N^2})$, $\lambda^T = (\lambda_1, \lambda_2, \dots, \lambda_{N^2})$

Eq. (2.6) shows that the matrix Q with the $(Q)_{ij} = q_{ij}$ is doubly stochastic matrix. The set of doubly stochastic matrices, Ω_n , is the convex hull of the permutation matrices (Birkhoff (1946), von Neumann (1953)). In other words, the doubly stochastic matrix, Ω_n , is the convex combination of the permutation matrices, P_n , that is

$$Q = \sum_i \tau_i P_i, \quad \sum_i \tau_i = 1, \quad \tau_i \geq 0 \quad \forall i \quad (2.8)$$

so, Eq.(2.7) takes form

$$\text{maximize} \quad \sum_i \tau_i \lambda^T P_i \eta, \quad \sum_i \tau_i = 1, \quad (2.9)$$

hence our problem reduces to a Linear Programming optimization over the convex set of feasible region. Here the feasible region is a simplex and its apex, yield when one of the τ_i equals 1 and the others equal zero, are the desired solutions of this optimization problem. This means that ρ and X have common eigenbasis.

3 Definition of 2^n -dimensional Bell-diagonal states

In order to put our discussion in a precise setting, let us first introduce 2^n -dimensional Bell-diagonal states acting on a bipartite system $H^A \otimes H^B$ with $\dim(H^A) = N = 2^n$ and $\dim(H^B) = N = 2^n$. To do this, let $S = \{g_1 = \gamma_1 \otimes \gamma_1, \dots, g_{2n} = \gamma_{2n} \otimes \gamma_{2n}\}$ be generated by $2n$ independent and commuting element such that $-I$ is not an element of S and $g_i^2 = I$ for all $g_i \in S$.

γ_j for $j = 1, 2, \dots, 2n+1$, known as Dirac matrices. (For a brief review about Dirac matrices and an explicit construction of γ_j , see Appendix II).

Hence, we can represent the density operators acting on a bipartite system $H^A \otimes H^B$ as:

$$\rho = \frac{1}{N^2} \sum_{i_1, i_2, \dots, i_{2n}} t_{i_1, i_2, \dots, i_{2n}} g_1^{i_1} g_2^{i_2} \dots g_{2n}^{i_{2n}} \quad (3.10)$$

where $i_1, i_2, \dots, i_{2n} \in \{0, 1\}$

Consider the projection operators $\{\pi_{i_1, i_2, \dots, i_{2n}}\}$ with

$$\pi_{i_1, i_2, \dots, i_{2n}} = \frac{1}{2^{2n}} \prod_{j=1}^{2n} (I + (-1)^{i_j} g_j) = \sum_{j_1, j_2, \dots, j_{2n}}^1 (-1)^{i_1 j_1 + i_2 j_2 + \dots + i_{2n} j_{2n}} g_1^{j_1} g_2^{j_2} \dots g_{2n}^{j_{2n}}$$

$$\pi_{i_1, i_2, \dots, i_{2n}} \pi_{j_1, j_2, \dots, j_{2n}} = \delta_{i_1 j_1} \dots \delta_{i_{2n} j_{2n}} \pi_{i_1, i_2, \dots, i_{2n}}, \quad \sum_{i_1, i_2, \dots, i_{2n}} \pi_{i_1, i_2, \dots, i_{2n}} = I, \quad (3.11)$$

then we get

$$\rho = \sum_{i_1, i_2, \dots, i_{2n}} p_{i_1, i_2, \dots, i_{2n}} \pi_{i_1, i_2, \dots, i_{2n}} \quad (3.12)$$

where

$$p_{i_1, i_2, \dots, i_{2n}} = \frac{1}{N^2} \sum_{j_1, j_2, \dots, j_{2n}} (-1)^{i_1 j_1 + i_2 j_2 + \dots + i_{2n} j_{2n}} t_{j_1, j_2, \dots, j_{2n}}$$

$$t_{j_1, j_2, \dots, j_{2n}} = \sum_{i_1, i_2, \dots, i_{2n}} (-1)^{i_1 j_1 + i_2 j_2 + \dots + i_{2n} j_{2n}} p_{i_1, i_2, \dots, i_{2n}} \quad (3.13)$$

From the theorem above it follows that the CSS states can be represented as:

$$\sigma = \frac{1}{N^2} \sum_{i_1, i_2, \dots, i_{2n}} \acute{t}_{i_1, i_2, \dots, i_{2n}} g_1^{i_1} g_2^{i_2} \dots g_{2n}^{i_{2n}} = \sum_{i_1, i_2, \dots, i_{2n}} \acute{p}_{i_1, i_2, \dots, i_{2n}} \pi_{i_1, i_2, \dots, i_{2n}} \quad (3.14)$$

and CCS states are

$$\chi_\rho = \frac{1}{N^2} \sum_{i_1, i_2, \dots, i_{2n}} \tilde{t}_{i_1, i_2, \dots, i_{2n}} g_1^{i_1} g_2^{i_2} \dots g_{2n}^{i_{2n}} = \sum_{i_1, i_2, \dots, i_{2n}} q_{i_1, i_2, \dots, i_{2n}} \pi_{i_1, i_2, \dots, i_{2n}}$$

$$\chi_\sigma = \frac{1}{N^2} \sum_{i_1, i_2, \dots, i_{2n}} \bar{t}_{i_1, i_2, \dots, i_{2n}} g_1^{i_1} g_2^{i_2} \dots g_{2n}^{i_{2n}} = \sum_{i_1, i_2, \dots, i_{2n}} \acute{q}_{i_1, i_2, \dots, i_{2n}} \pi_{i_1, i_2, \dots, i_{2n}} \quad (3.15)$$

4 Calculation of classical states and quantum discord

First of all, we note that the result of above theorem can straightforwardly be used to obtain the CCS states of 2^n -dimensional Bell-diagonal states. To do this recall that the Pauli operators on a single qubit are $\{I, \sigma_x, \sigma_y, \sigma_z\}$. The representation of the Pauli group we will deal with is the group formed by elements of the form $G_n = \{i^k P_1 \otimes P_2 \otimes \dots \otimes P_n\}$ where each P_i is an element of $\{I, \sigma_x, \sigma_y, \sigma_z\}$.

Suppose Γ is a subgroup of G_n generated by elements $\{\Gamma_{i_1 i_2 \dots i_n j_1 j_2 \dots j_n}\}$. There is an extremely useful way of presenting the generators of Γ [40]. To do this, we use $r(\Gamma_{i_1 i_2 \dots i_n j_1 j_2 \dots j_n}) = [i_1 i_2 \dots i_n | j_1 j_2 \dots j_n]$ to denote the $2n$ -dimensional row vector representation of an element of the Γ . The left hand side of the row vector contains 1s to indicate which generators contain σ_x s, and the right hand side contains 1s to indicate which generators contain σ_z s; the presence of a 1 on both sides indicates a σ_y in the generator. More explicitly, it is constructed as follows. If γ_i contains an I on the j th qubit then the j th and $n+j$ th column elements are 0; if it contains an σ_x on the j th qubit then the j th column element is a 1 and the $n+j$ th column element is a 0; if it contains a σ_z on the j th qubit then the j th column element is 0 and the $n+j$ th column element is 1; if it contains a σ_y on the j th qubit then both the j th and $n+j$ th columns are 1. Let us define a $2n \times 2n$ matrix Λ by

$$\begin{pmatrix} 0 & I_{n \times n} \\ I_{n \times n} & 0 \end{pmatrix}, \quad (4.16)$$

then the elements $\Gamma_{i_1 i_2 \dots i_n j_1 j_2 \dots j_n}$ and $\Gamma_{i_1' i_2' \dots i_n' j_1' j_2' \dots j_n'}$ are easily seen to commute if and only if

$$r(\Gamma_{i_1 i_2 \dots i_n j_1 j_2 \dots j_n}) \Lambda r(\Gamma_{i_1' i_2' \dots i_n' j_1' j_2' \dots j_n'}) = 0 \pmod{2}. \quad (4.17)$$

Let $g_{c_k} \in \{\Gamma_{i_1 i_2 \dots i_n j_1 j_2 \dots j_n} \otimes \Gamma_{i_1' i_2' \dots i_n' j_1' j_2' \dots j_n'}\}$ for $k = 1, \dots, n$, such that the Eq.(4.17) is satisfied, then we can rewrite the state ρ such as:

$$\rho = \frac{1}{N^2} \sum_{i_1, \dots, i_{2n}} t_{i_1, \dots, i_{2n}} g_{c_1}^{i_1} \dots g_{c_n}^{i_n} g_{n+1}^{i_{n+1}} \dots g_{2n}^{i_{2n}}.$$

Since classical states contains mixtures of locally distinguishable states hence we can rewrite the expression of the CCS of ρ such as:

$$\chi = \frac{1}{N^2} \sum_{i_1, \dots, i_{2n}} \tilde{t}_{i_1, i_2, \dots, i_n, \underbrace{0, 0, \dots, 0}_{n \text{ fold}}} g_{c_1}^{i_1} \dots g_{c_n}^{i_n} g_{n+1}^0 \dots g_{2n}^0 = \frac{1}{N^2} \sum_{i_1, \dots, i_{2n}} \tilde{t}_{i_1, i_2, \dots, i_n, \underbrace{0, 0, \dots, 0}_{n \text{ fold}}} g_{c_1}^{i_1} \dots g_{c_n}^{i_n} \quad (4.18)$$

where $\underbrace{t_{0, \dots, 0}}_{n \text{ fold}} = \underbrace{\tilde{t}_{0, \dots, 0}}_{n \text{ fold}} = 1$.

Here we show that some of $\tilde{t}_{i_1, i_2, \dots, i_n, \underbrace{0, 0, \dots, 0}_{n \text{ fold}}}$ are zero and rest of them are the same of

$t_{i_1, i_2, \dots, i_n, \underbrace{0, 0, \dots, 0}_{n \text{ fold}}}$. To show this note that the eigenvalues of χ which are 2^n -fold degenerate, are given by

$$q_{i_1, \dots, i_n, i_{n+1}, \dots, i_{2n}} = \frac{1}{N^2} \sum_{j_1, \dots, j_n} (-1)^{i_1 j_1 + \dots + i_n j_n} \tilde{t}_{j_1, j_2, \dots, j_n, \underbrace{0, 0, \dots, 0}_{n \text{ fold}}}, \quad (4.19)$$

so the problem of finding the CCS to ρ is reduced to the problem

$$\begin{cases} \text{minimize} & S(\rho||\chi) \\ \text{subject to} & \sum_{i_1, \dots, i_n, i_{n+1}, \dots, i_{2n}} q_{i_1, \dots, i_n, i_{n+1}, \dots, i_{2n}} = 1. \end{cases} \quad (4.20)$$

The dual Lagrangian associated with this problem, is given by

$$\begin{aligned} L = \sum_{i_1, \dots, i_{2n}} p_{i_1, \dots, i_{2n}} \log p_{i_1, \dots, i_{2n}} - \sum_{i_1, \dots, i_n} \left(\sum_{i_{n+1}, \dots, i_{2n}} p_{i_1, \dots, i_n, i_{n+1}, \dots, i_{2n}} \right) \log q_{i_1, \dots, i_n, i_{n+1}, \dots, i_{2n}} \\ + \mu \left[\sum_{i_1, \dots, i_n, i_{n+1}, \dots, i_{2n}} q_{i_1, \dots, i_n, i_{n+1}, \dots, i_{2n}} - 1 \right]. \end{aligned} \quad (4.21)$$

By calculating the gradient of the dual Lagrangian with respect to $\tilde{t}_{j_1, j_2, \dots, j_n, \underbrace{0, 0, \dots, 0}_{n \text{ fold}}}$ and making it zero we get

$$\frac{\partial L}{\partial \tilde{t}_{j_1, j_2, \dots, j_n, \underbrace{0, 0, \dots, 0}_{n \text{ fold}}}} = - \sum_{i_1, \dots, i_n} \left[\frac{\sum_{i_{n+1}, \dots, i_{2n}} p_{i_1, \dots, i_n, i_{n+1}, \dots, i_{2n}}}{q_{i_1, \dots, i_n, i_{n+1}, \dots, i_{2n}}} + \mu \right] (-1)^{i_1 j_1 + \dots + i_n j_n} = 0. \quad (4.22)$$

Since

$$\sum_{i_{n+1}, \dots, i_{2n}} p_{i_1, \dots, i_n, i_{n+1}, \dots, i_{2n}} = \frac{2^n}{N^2} \sum_{j_1, j_2, \dots, j_{2n}} (-1)^{i_1 j_1 + \dots + i_n j_n} t_{j_1, j_2, \dots, j_n, \underbrace{0, 0, \dots, 0}_{n \text{ fold}}}, \quad (4.23)$$

then using Eq. (4.22) one can show that

$$\tilde{t}_{j_1, j_2, \dots, j_n, \underbrace{0, 0, \dots, 0}_{n \text{ fold}}} = t_{j_1, j_2, \dots, j_n, \underbrace{0, 0, \dots, 0}_{n \text{ fold}}} \quad \forall j_1, \dots, j_n, \quad (4.24)$$

then we can rewrite the CCS of ρ such as:

$$\chi = \frac{1}{N^2} \sum_{i_1, \dots, i_{2n}} t_{i_1, i_2, \dots, i_n, \underbrace{0, 0, \dots, 0}_{n \text{ fold}}} g_{c_1}^{i_1} \dots g_{c_n}^{i_n} \quad (4.25)$$

In the rest of this section we calculate the quantum discord of 2^{2n} -dimensional Bell-diagonal states. Note that the key difference between the original definition of discord [11, 12] and the definition in Eq.(2.2) is in minimization. We minimize the quantity D , while for the original discord, $D - L_\rho$ is minimized [19] where $L_\rho = S(\pi_{\chi_\rho}) - S(\pi_\rho)$. Since for the Bell-diagonal states $L_\rho = 0$, hence the two forms of discord are the same. Now, using Eq.(4.24) we give an exact analytical formula quantum discord for 2^n -dimensional Bell-diagonal states such as:

$$D = \sum_{i_1, i_2, \dots, i_{2n}} p_{i_1, i_2, \dots, i_{2n}} \log p_{i_1, i_2, \dots, i_{2n}} - \max \frac{1}{N^2} \sum_{i_1, i_2, \dots, i_n} \sum_{j_1, \dots, j_n} (-1)^{i_1 j_1 + \dots + i_n j_n} t_{j_1, j_2, \dots, j_n, \underbrace{0, 0, \dots, 0}_{n \text{ fold}}} \log \frac{1}{N^2} \sum_{j_1, \dots, j_n} (-1)^{i_1 j_1 + \dots + i_n j_n} t_{j_1, j_2, \dots, j_n, \underbrace{0, 0, \dots, 0}_{n \text{ fold}}} \quad (4.26)$$

where the maximum is taken over all parameters $\{t_{j_1, j_2, \dots, j_n, \underbrace{0, 0, \dots, 0}_{n \text{ fold}}}\}$. Here we should mention that the set $\{g_{c_1}^{i_1} \dots g_{c_n}^{i_n}\}$ in Eq.(4.25) can be chosen in many different ways. Since in the optimum strategy D is at its minimum, then we chose the set $\{g_{c_1}^{i_1} \dots g_{c_n}^{i_n}\}$ or equivalently the parameters $\{t_{j_1, j_2, \dots, j_n, \underbrace{0, 0, \dots, 0}_{n \text{ fold}}}\}$ such that the second part in the Eq.(4.26) is maximized. To give an intuitive understanding of this subject, let us illustrate it by a fundamental examples.

4.1 Example 1: Generalized bloch sphere states

Using Eq.(3.10) the generalized bloch sphere states [41] are given by

$$\rho = \frac{1}{N^2} [I + \sum_{k=1}^{2n} t_{0, \dots, \underbrace{1}_k, \dots, 0} g_k + t_{1, 1, \dots, 1} g_{2n+1}] \quad (4.27)$$

where $g_{2n+1} = g_1 g_2 \dots g_{2n}$. Since classical states contains mixtures of locally distinguishable states hence using Eq.(4.25) one can conclude that the CCS of generalized bloch sphere states lie on the Cartesian axes. That is only one of the $\{t_{i_1, i_2, \dots, i_n, \underbrace{0, 0, \dots, 0}_{n \text{ fold}}}\}$ is nonzero. We assume that, without loss of generality, $t_{1, 0, \dots, 0} \neq 0$, then the CCS of generalized bloch sphere states

are given by

$$\chi = \frac{1}{N^2} [I + t_{1,0,\dots,0} g_1], \quad (4.28)$$

In the optimum strategy $S(\rho||\chi)$ is at its minimum, that is we have $t_{1,0,\dots,0} = t_{max}$. Hence using, (4.26), we obtain

$$D = \sum_{i_1, i_2, \dots, i_{2n}}^{N^2-1} p_{i_1, i_2, \dots, i_{2n}} \log p_{i_1, i_2, \dots, i_{2n}} - \frac{1-t_{max}}{2} \log(1-t_{max}) - \frac{1+t_{max}}{2} \log(1+t_{max}) + 2 \log(N). \quad (4.29)$$

This is in agreement with the result obtained in [41] for $N=M$.

4.2 Example 2: 2²-dimensional Bell-diagonal states

As the second example, to keep our discussion simple, let us focus attention on $n=2$ case. In this case, using (4.25) we have

$$\chi = \frac{1}{16} \sum_{i_1, \dots, i_4} t_{i_1, i_2, 0, 0} g_{c_1}^{i_1} g_{c_2}^{i_2} \quad (4.30)$$

where, using (4.17) we get

$$\begin{aligned} (g_{c_1}, g_{c_2}) \in \{ & (\Gamma_{1000} \otimes \Gamma_{1000}, \Gamma_{0101} \otimes \Gamma_{0101}), (\Gamma_{0001} \otimes \Gamma_{0001}, \Gamma_{0110} \otimes \Gamma_{0110}), (\Gamma_{1000} \otimes \Gamma_{1000}, \Gamma_{0110} \otimes \Gamma_{0110}), \\ & (\Gamma_{0100} \otimes \Gamma_{0100}, \Gamma_{0011} \otimes \Gamma_{0011}), (\Gamma_{1111} \otimes \Gamma_{1111}, \Gamma_{1100} \otimes \Gamma_{1100}), (\Gamma_{1111} \otimes \Gamma_{1111}, \Gamma_{1010} \otimes \Gamma_{1010}), \\ & (\Gamma_{1000} \otimes \Gamma_{1000}, \Gamma_{0011} \otimes \Gamma_{0011}), (\Gamma_{0010} \otimes \Gamma_{0010}, \Gamma_{0101} \otimes \Gamma_{0101}), (\Gamma_{1111} \otimes \Gamma_{1111}, \Gamma_{1001} \otimes \Gamma_{1001}), \\ & (\Gamma_{0010} \otimes \Gamma_{0010}, \Gamma_{1001} \otimes \Gamma_{1011}), (\Gamma_{0100} \otimes \Gamma_{0100}, \Gamma_{1010} \otimes \Gamma_{1010}), (\Gamma_{0001} \otimes \Gamma_{0001}, \Gamma_{1100} \otimes \Gamma_{1100}) \\ & (\Gamma_{0010} \otimes \Gamma_{0010}, \Gamma_{1100} \otimes \Gamma_{1100}), (\Gamma_{0100} \otimes \Gamma_{0100}, \Gamma_{1001} \otimes \Gamma_{1001}), (\Gamma_{0001} \otimes \Gamma_{0001}, \Gamma_{1010} \otimes \Gamma_{1010}) \}, \end{aligned} \quad (4.31)$$

and

$$\begin{aligned} \Gamma_{1000} &= \sigma_x \otimes I, \Gamma_{0100} = i\sigma_y \otimes \sigma_x, \Gamma_{0010} = i\sigma_y \otimes \sigma_y, \Gamma_{0001} = i\sigma_y \otimes \sigma_z, \Gamma_{1111} = -\sigma_z \otimes I, \\ \Gamma_{1100} &= -\sigma_z \otimes \sigma_x, \Gamma_{1010} = -\sigma_z \otimes \sigma_y, \Gamma_{1001} = i\sigma_z \otimes \sigma_z, \Gamma_{0110} = -iI \otimes \sigma_z, \Gamma_{0101} = I \otimes \sigma_y, \\ \Gamma_{0011} &= -I \otimes \sigma_x, \Gamma_{0111} = \sigma_x \otimes I, \Gamma_{1011} = \sigma_x \otimes \sigma_x, \Gamma_{1101} = \sigma_x \otimes \sigma_y, \Gamma_{1110} = \sigma_x \otimes \sigma_z. \end{aligned} \quad (4.32)$$

Let us choose $g_{c_1} = \Gamma_{1000} \otimes \Gamma_{1000}$ and $g_{c_2} = \Gamma_{0101} \otimes \Gamma_{0101}$, hence we have

$$\chi = \frac{1}{16}(I + t_{1000}\Gamma_{1000} \otimes \Gamma_{1000} + t_{0100}\Gamma_{0101} \otimes \Gamma_{0101} + t_{1100}\Gamma_{1101} \otimes \Gamma_{1101}) \quad (4.33)$$

Then, using (4.26) quantum discord is given by

$$D = \sum_{i_1, i_2, i_3, i_4} p_{i_1, i_2, i_3, i_4} \log p_{i_1, i_2, i_3, i_4} - \sum_{i=1}^4 p_{q_i} \log p_{q_i}. \quad (4.34)$$

where

$$\begin{aligned} p_{q_1} &= \frac{1}{16}[1 + t_{1000} + t_{0100} + t_{1100}] \\ p_{q_2} &= \frac{1}{16}[1 + t_{1000} - t_{0100} - t_{1100}] \\ p_{q_3} &= \frac{1}{16}[1 - t_{1000} + t_{0100} - t_{1100}] \\ p_{q_4} &= \frac{1}{16}[1 - t_{1000} - t_{0100} + t_{1100}] \end{aligned} \quad (4.35)$$

5 Subadditivity of correlations of a quantum state

It has been conjectured [19] that the correlations of a quantum state are subadditive in the sense $T_\rho \geq E + Q + C_\sigma$ (where T_ρ is total mutual information which defined as $S(\rho||\pi_\rho)$ and C_σ is the classical correlation $S(\chi_\sigma|\pi_\sigma)$). In general, from an analytical point of view, the derivation of closed expressions of relative entropy of entanglement involves optimization procedures that are very complicated to perform. Hence, here we consider the inverse problem [38] and investigate additivity relations between different correlations for separable generalized bloch sphere states. These states are given by

$$\sigma = \frac{1}{N^2}[I + \sum_{k=1}^{2n} \acute{t}_{0, \dots, \underbrace{1}_k, \dots, 0} g_k + \acute{t}_{1, 1, \dots, 1} g_{2n+1}], \quad (5.36)$$

with the eigenvalues

$$\acute{p}_{i_1, i_2, \dots, i_{2n}} = \frac{1}{N^2}[1 + \sum_{k=1}^{2n} (-1)^{i_k} \acute{t}_{0, \dots, \underbrace{1}_k, \dots, 0} + (-1)^n (-1)^{i_1 + \dots + i_{2n}} \acute{t}_{1, 1, \dots, 1}] \quad (5.37)$$

where $g_{2n+1} = g_1 g_2 \dots g_{2n}$. The separable generalized bloch sphere states are actually bounded by $\sum_{k=1}^{2n} |\acute{t}_{0, \dots, \underbrace{1}_k, \dots, 0}| + |\acute{t}_{1, 1, \dots, 1}| \leq 1$ or, equivalently, $\acute{p}_{i_1, i_2, \dots, i_{2n}} \leq \frac{2}{N^2}$ ($\forall i_1, i_2, \dots, i_{2n}$). The

family of all entangled states, $\rho(x, \sigma)$, for which σ is the CSS is given by [39]

$$\rho(x, \sigma) = \sigma - x L_\sigma^{-1}(w_{i_1, i_2, \dots, i_{2n}}), \quad 0 < x \leq x_{max} \quad (5.38)$$

where L_α is linear operator. In the eigenbasis of α , $\alpha = \text{diag}(a_1, \dots, a_n)$ is a diagonal matrix, where $a_1, \dots, a_n > 0$ and for any $\beta = [b_{i,j=1}^n]$, $L_\alpha(\beta)$ is defined by

$$[L_\alpha(\beta)]_{kl} = \begin{cases} b_{kl} \frac{\ln a_k - \ln a_l}{a_k - a_l}, & \text{if } a_k \neq a_l \\ b_{kl} \frac{1}{a}, & \text{if } a_k = a_l = a. \end{cases} \quad (5.39)$$

$w_{i_1, i_2, \dots, i_{2n}}$ s are entanglement witnesses (EW) of 2^n -dimensional Bell-diagonal states. Here, x_{max} is defined such that $\rho(x_{max}, \sigma) \in H_{n,+1}$ (convex set of positive hermitian matrices of trace one) and $\rho(x_{max}, \sigma)$ has at least one zero eigenvalue. We also note that $w_{i_1, i_2, \dots, i_{2n}}$ is normalized; i.e. $\text{Tr}(w_{i_1, i_2, \dots, i_{2n}})^2 = 1$ and $\text{Tr}(L_\sigma^{-1}(w_{i_1, i_2, \dots, i_{2n}})) = 0$. In general EW of the 2^{2n} -dimensional Bell-diagonal states is given by [42]

$$w_{i_1, i_2, \dots, i_{2n}} = \frac{1}{N \sqrt{2(n+1)}} [I_{2^{2n}} + \sum_{k=1}^{2n} (-1)^{i_k} \gamma_k^{2n} \otimes \gamma_k^{2n} - (-i)^{2n} (-1)^{i_1+i_2+\dots+i_{2n}} \gamma_{2n+1}^{2n} \otimes \gamma_{2n+1}^{2n}], \quad (5.40)$$

with the eigenvalue

$$\lambda_{j_1, j_2, \dots, j_{2n}}^{w_{i_1, i_2, \dots, i_{2n}}} = \frac{1}{N \sqrt{2(n+1)}} [1 + \sum_{k=1}^{2n} (-1)^{i_k} (-1)^{j_k} - (-1)^{i_1+i_2+\dots+i_{2n}+j_1+j_2+\dots+j_{2n}}]. \quad (5.41)$$

Now define the real symmetric matrix [39]

$$[S(\sigma)]_{k_1 \dots k_{2n}, l_1 \dots l_{2n}} = \frac{\dot{p}_{k_1 \dots k_{2n}} - \dot{p}_{l_1 \dots l_{2n}}}{\ln \dot{p}_{k_1 \dots k_{2n}} - \ln \dot{p}_{l_1 \dots l_{2n}}}, \quad (5.42)$$

hence we get

$$S(\sigma) = \sum_{k_1 \dots k_{2n} \neq l_1 \dots l_{2n}} \frac{\dot{p}_{k_1 \dots k_{2n}} - \dot{p}_{l_1 \dots l_{2n}}}{\ln \dot{p}_{k_1 \dots k_{2n}} - \ln \dot{p}_{l_1 \dots l_{2n}}} \Pi_{k_1 \dots k_{2n}} \Pi_{l_1 \dots l_{2n}} + \sum_{k_1 \dots k_{2n}} \dot{p}_{k_1 \dots k_{2n}} \Pi_{k_1 \dots k_{2n}}. \quad (5.43)$$

Note that L_σ is an invertible operator, where $L_\sigma^{-1}(w_{i_1, i_2, \dots, i_{2n}}) = w_{i_1, i_2, \dots, i_{2n}} \bullet S(\sigma)$ where $A \bullet B$ is the entrywise product of two matrices of A and B. Let

$$w_{i_1, i_2, \dots, i_{2n}} = \sum_{j_1 \dots j_{2n}} \lambda_{j_1 \dots j_{2n}}^{i_1 \dots i_{2n}} \Pi_{j_1 \dots j_{2n}} \quad (5.44)$$

then, using (5.38,5.42,5.43), we obtain

$$p_{j_1,j_2,\dots,j_{2n}} = \acute{p}_{j_1,j_2,\dots,j_{2n}} [1 - x \lambda_{j_1,j_2,\dots,j_{2n}}^{w_{i_1,i_2,\dots,i_{2n}}}] . \quad (5.45)$$

We now focus our attention on the subadditivity of correlations. By direct calculation one gets

$$\begin{aligned} E + Q + C_\sigma - T_\rho &= \sum p_{j_1,j_2,\dots,j_{2n}} \log \frac{p_{j_1,j_2,\dots,j_{2n}}}{\acute{p}_{j_1,j_2,\dots,j_{2n}}} + \sum \acute{p}_{j_1,j_2,\dots,j_{2n}} \log \frac{\acute{p}_{j_1,j_2,\dots,j_{2n}}}{\acute{q}_{j_1,j_2,\dots,j_{2n}}} \\ &+ \sum \acute{q}_{j_1,j_2,\dots,j_{2n}} \log \acute{q}_{j_1,j_2,\dots,j_{2n}} - \sum p_{j_1,j_2,\dots,j_{2n}} \log p_{j_1,j_2,\dots,j_{2n}} = \sum (\acute{p}_{j_1,j_2,\dots,j_{2n}} - p_{j_1,j_2,\dots,j_{2n}}) \log \acute{p}_{j_1,j_2,\dots,j_{2n}}, \end{aligned} \quad (5.46)$$

then, using (5.45), we have

$$E + Q + C_\sigma - T_\rho = x \sum_{j_1,j_2,\dots,j_{2n}} \lambda_{j_1,j_2,\dots,j_{2n}}^{\phi_{i_1,i_2,\dots,i_{2n}}} \acute{p}_{j_1,j_2,\dots,j_{2n}} \log \acute{p}_{j_1,j_2,\dots,j_{2n}} \quad (5.47)$$

Assume, with no loss of generality, the $w_{0,0,\dots,0}$, then we have $\acute{p}_{1,1,\dots,1} = \frac{2}{N^2}$, that is

$$\sum_{j=1}^{2n} \acute{t}_{0,\dots,\underbrace{1}_j,\dots,0} - (-1)^n \acute{t}_{1,1,\dots,1} + 1 = 0. \quad (5.48)$$

Thus, we have the following optimization problem

$$\max \quad x \sum_{j_1,j_2,\dots,j_{2n}} \lambda_{j_1,j_2,\dots,j_{2n}}^{w_{0,0,\dots,0}} \acute{p}_{j_1,j_2,\dots,j_{2n}} \log \acute{p}_{j_1,j_2,\dots,j_{2n}} \quad (5.49)$$

$$\text{subject to} \begin{cases} \sum_{j=1}^{2n} (-1)^{i_j} \acute{t}_{0,\dots,\underbrace{1}_j,\dots,0} + (-1)^{i_{2n+1}} \acute{t}_{1,1,\dots,1} \leq 1 \\ \sum_{j=1}^{2n} \acute{t}_{0,\dots,\underbrace{1}_j,\dots,0} - (-1)^n \acute{t}_{1,1,\dots,1} + 1 = 0 \end{cases} \quad (5.50)$$

The dual Lagrangian associated with this problem, is given by

$$\begin{aligned} L &= x \sum_{j_1,j_2,\dots,j_{2n}} \lambda_{j_1,j_2,\dots,j_{2n}}^{\phi_{i_1,i_2,\dots,i_{2n}}} \acute{p}_{j_1,j_2,\dots,j_{2n}} \log \acute{p}_{j_1,j_2,\dots,j_{2n}} + \mu_{0,\dots,0} [\sum_{j=1}^{2n} \acute{t}_{0,\dots,\underbrace{1}_j,\dots,0} - (-1)^n \acute{t}_{1,1,\dots,1} + 1] \\ &+ \sum_{i_1,\dots,i_{2n+1}} \mu_{i_1,\dots,i_{2n+1}} [\sum_{j=1}^{2n} (-1)^{i_j} \acute{t}_{0,\dots,\underbrace{1}_j,\dots,0} + (-1)^{i_{2n+1}} \acute{t}_{1,1,\dots,1} - 1], \end{aligned} \quad (5.51)$$

and the complementary slackness condition(see the Appendix I) is given by

$$\mu_{i_1, \dots, i_{2n+1}} \left[\sum_{j=1}^{2n} (-1)^{i_j} \acute{t}_{0, \dots, \underbrace{1}_j, \dots, 0} + (-1)^{i_{2n+1}} \acute{t}_{1, 1, \dots, 1} - 1 \right] = 0 \quad (5.52)$$

The possible optimal solutions for above problem are the edge and vertices solutions. First of all, we consider vertex solution that is one of the vertices. In the case of odd n, we have

$$\left\{ \begin{array}{l} \acute{t}_{1, 1, \dots, 1} = -1 \\ \acute{t}_{0, \dots, \underbrace{1}_j, \dots, 0} = 0, \quad \forall j \in \{1, \dots, 2n\} \end{array} \right. \quad (5.53)$$

then $\acute{p}_{j_1, j_2, \dots, j_{2n}} = 0$ if $j_1 + j_2 + \dots + j_{2n} = \text{odd}$ and $\acute{p}_{j_1, j_2, \dots, j_{2n}} = \frac{2}{N^2}$ if $j_1 + j_2 + \dots + j_{2n} = \text{even}$.

Hence Eq.(5.47) gives

$$E + Q + C_\sigma - T_\rho = -\frac{(2n-1)x}{2^{3n-1}\sqrt{2(n+1)}} \log \frac{2}{N^2} \sum_{j_1, j_2, \dots, j_{2n}} \left[1 + \sum_{k=1}^{2n} (-1)^{j_k} - (-1)^{j_1 + j_2 + \dots + j_{2n}} \right] = 0 \quad (5.54)$$

Similarly, for other vertices(say $\acute{t}_{1, 1, \dots, 1} = -1, \acute{t}_{0, \dots, \underbrace{1}_j, \dots, 0} = 0, \quad \forall j \in \{1, \dots, 2n\}$ (even n) and , with no loss of generality, $\acute{t}_{1, 0, \dots, 0} = -1, \acute{t}_{1, 1, \dots, 1} = \acute{t}_{0, \dots, \underbrace{1}_j, \dots, 0} = 0, \quad \forall j \in \{2, \dots, 2n\}$) one can show that $E + Q + C_\sigma - T_\rho = 0$. Let us next turn our attention to the edge solutions. In this case we have

$$\begin{aligned} L = & \sum_{i_1, \dots, i_{2n}} p_{i_1, \dots, i_{2n}} \log p_{i_1, \dots, i_{2n}} - \sum_{i_1, \dots, i_{2n}} p_{i_1, \dots, i_{2n}} \log \acute{p}_{i_1, \dots, i_{2n}} \\ & + \mu_{0, \dots, 0} [\acute{t}_{10\dots 0} + \acute{t}_{010\dots 0} + \dots + \acute{t}_{00\dots 1} - (-1)^n \acute{t}_{11\dots 1} + 1] \end{aligned} \quad (5.55)$$

By calculating the gradient of the dual Lagrangian with respect to making it zero one can show that

$$\acute{t}_{10\dots 0} = \acute{t}_{010\dots 0} = \dots = \acute{t}_{00\dots 1}. \quad (5.56)$$

Hence, using (5.37,5.48,5.56), one finds that

$$E + Q + C_\sigma - T_\rho = \frac{1}{2^{3n}\sqrt{2(2n+1)}} \sum_{i_1 \dots i_{2n}} \left[1 + \sum_{k=1}^{2n} (-1)^{i_k} - (-1)^{i_1 + i_1 + \dots + i_{2n}} \right] [1 + (-1)^{i_1 + i_1 + \dots + i_{2n}} +$$

$$\left(\sum_{k=1}^{2n} (-1)^{i_k} + 2n(-1)^{i_1+i_1+\dots+i_n}\right) \acute{t}_{10\dots 0} \log \frac{1}{2^{2n}} [1 + (-1)^{i_1+i_1+\dots+i_{2n}} + \left(\sum_{k=1}^{2n} (-1)^{i_k} + 2n(-1)^{i_1+i_1+\dots+i_n}\right) \acute{t}_{10\dots 0}] \quad (5.57)$$

The above function is convex and it is zero at $\acute{t}_{10\dots 0} = 0$, then one can immediately deduce that $E + Q + C_\sigma - T_\rho$ becomes negative for the acceptable value of $\acute{t}_{10\dots 0}$. In general, the Eq.(5.47) is the difference of two convex functions, which both of them are non-positive and the intersection points of these two functions are in the vertices of the feasible region. On the other hand, for the edge and vertices solutions $E + Q + C_\sigma - T_\rho \leq 0$. Then one can conclude that the correlations of generalized bloch sphere states are subadditive in the feasible region.

6 Conclusion

In the unified view of quantum and classical correlations the quantifications are done by the relative entropy and optimization of relative entropy is known to be a difficult problem. In this work, we have presented a general algorithm via exact convex optimization to the problem of finding CSS and CCS for a given entanglement state ρ . Using the obtained CCS for the 2^n -dimensional Bell-diagonal states, we have derived analytical formula for the quantum discord of these states. As illustrating examples, we have analyzed the case of the separable generalized bloch sphere states and 2^2 -dimensional Bell-diagonal states and described how to apply the formula for this cases. We have also shown that the separable generalized bloch sphere states is subadditive. While our analysis is for a special states of bipartite quantum system, it serves to provide a unified explanation for a variety of states. In fact, this approach is completely general and could be applied for multipartite states in all dimensions. The main conclusion is that the presented algorithm provide indispensable prerequisites for further investigation and can bring a robustness in constructing CSS and CCS for a given multipartite states in all dimensions. Application of this algorithm to other quantum system and finding related CSS and CCS is still an open problem which is under investigation.

Appendix I:

Convex optimization review: An optimization problem [40], has the standard form

$$\begin{cases} \text{maximize } f_0(x). \\ \text{subject to } f_i(x) \leq b_i, i = 1, \dots, m \quad h_i(x) = 0, i = 1, \dots, p \end{cases} \quad (\text{I-1})$$

Where the vector $x = (x_1, \dots, x_n)$ is the optimization variable of the problem, the function $f_0 : R^n \rightarrow R$ is the objective function, the functions $f_i : R^n \rightarrow R, i = 1, \dots, m$ are the (inequality) constraint functions, and the constants b_1, \dots, b_m are the limits, or bounds, for the constraints. A convex optimization problem, is an optimization problem where the objective and the constraint functions are convex functions which means they satisfy inequality $f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y)$, for all $(x, y, \alpha, \beta) \in R$ with $\alpha + \beta = 1, \alpha \geq 0, \beta \geq 0$ and the equality constraint functions $h_i(x) = 0$ must be affine (A set $C \in R^n$ is affine if the line through any two distinct points in C lies in C). One can solve this convex optimization problem using Lagrangian duality. The basic idea in the Lagrangian duality is to take the constraints in convex optimization problem into account by augmenting the objective function with a weighted sum of the constraint functions. The Lagrangian $L : R^n \times R^m \times R^p \rightarrow R$ associated with the problem is defined as

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x). \quad (\text{I-2})$$

The Lagrange dual function $g : R^n \times R^m \times R^p \rightarrow R$ is defined as the minimum value of the Lagrangian over x : for $\lambda \in R^m, \nu \in R^p$,

$$g(\lambda, \nu) = \inf_{x \in D} L(x, \lambda, \nu). \quad (\text{I-3})$$

The dual function yields lower bounds on the optimal value p^* of the convex optimization problem, i.e for any $\lambda \geq 0$ and any ν we have

$$g(\lambda, \nu) \leq p^*. \quad (\text{I-4})$$

The optimal value of the Lagrange dual problem, which we denote d^* , is, by definition, the best lower bound on d^* that can be obtained from the Lagrange dual function. In particular, we have the simple but important inequality

$$d^* \leq p^*. \quad (\text{I-5})$$

This property is called weak duality. If the equality $d^* = p^*$ holds, i.e., the optimal duality gap is zero, then we say that strong duality holds. If strong duality holds and a dual optimal solution (λ^*, ν^*) exists, then any primal optimal point is also a minimizer of $L(x, \lambda^*, \nu^*)$. This fact sometimes allows us to compute a primal optimal solution from a dual optimal solution. For the best lower bound that can be obtained from the Lagrange dual function one can solve the following optimization problem

$$\begin{cases} \text{maximize} & g(\lambda, \nu). \\ \text{subject to} & \lambda \geq 0 \end{cases} \quad (\text{I-6})$$

This problem is called the Lagrange dual problem associated with the main problem. Conditions for the optimality of a convex problem is called Karush-Kuhn-Tucker (KKT) conditions. If f_i are convex and h_i are affine, and $\tilde{x}, \tilde{\lambda}, \tilde{\nu}$ are any points that satisfy the KKT conditions

$$\begin{aligned} h_i(\tilde{x}) &= 0, \quad i = 1, \dots, p, \\ f_i(\tilde{x}) &\leq 0, \quad i = 1, \dots, m, \\ \tilde{\lambda}_i &\geq 0 \quad \tilde{\lambda}_i f_i(\tilde{x}) = 0, \quad i = 1, \dots, m, \\ \nabla f_0(\tilde{x}) + \sum_i^m \tilde{\lambda}_i \nabla f_i(\tilde{x}) + \sum_i^p \tilde{\nu}_i \nabla h_i(\tilde{x}) &= 0. \end{aligned} \quad (\text{I-7})$$

then \tilde{x} and $(\tilde{\lambda}, \tilde{\nu})$ are primal and dual optimal, with zero duality gap. In other words, for any convex optimization problem with differentiable objective and constraint functions, any points that satisfy the KKT conditions are primal and dual optimal, and have zero duality gap. Hence, $f_0(\tilde{x}) = g(\tilde{\lambda}, \tilde{\nu})$.

The condition $\tilde{\lambda}_i f_i(\tilde{x}) = 0$, $i = 1, \dots, m$, is known as complementary slackness; it holds for any primal optimal \tilde{x} and any dual optimal $(\tilde{\lambda}, \tilde{\nu})$ (when strong duality holds)

Appendix II:

Throughout the paper, we have used the formalism of Dirac γ matrices. Therefore, in this appendix we define the algebra of Dirac γ matrices and exhibit matrices which realize the algebra in the Euclidean representation and explain our notations and conventions.

To do this, let γ_μ , $\mu = 1, \dots, d$, be a set of d matrices satisfying the anticommuting relations:

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu} I, \quad (\text{I-1})$$

in which I is the identity matrix. These matrices are the generators of a Clifford algebra similar to the algebra of operators acting on Grassmann algebras. It follows from relations (I-1) that the γ matrices generate an algebra which, as a vector space, has a dimension 2^d . In the following, we will give an inductive construction ($d \rightarrow d + 2$) of hermitian matrices satisfying (I-1). In the algebra one element plays a special role, the product of all γ matrices. The matrix γ_s :

$$\gamma_s = i^{\frac{-d}{2}} \gamma_1 \gamma_2 \dots \gamma_d, \quad (\text{I-2})$$

anticommutes, because d is even, with all other γ matrices and $\gamma_s^2 = I$.

In calculations involving γ matrices, it is not always necessary to distinguish γ_s from other γ matrices. Identifying thus γ_s with γ_{d+1} , we have:

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij} I, i, j = 1, \dots, d, d + 1. \quad (\text{I-3})$$

The Greek letters $\mu\nu\dots$ are usually used to indicate that the value $d + 1$ for the index has been excluded.

An explicit construction of $\gamma_i^{(d)}$

It is sometimes useful to have an explicit realization of the algebra of γ matrices. For $d = 2$, the standard Pauli matrices realize the algebra:

$$\begin{aligned}\gamma_1^{(d=2)} = \sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma_2^{(d=2)} = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\ \gamma_s^{(d=2)} = \gamma_3^{(d=2)} = \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\end{aligned}\tag{I-4}$$

The three matrices are hermitian, i.e., $\gamma_i = \gamma_i^\dagger$. The matrices γ_1 and γ_3 are symmetric and γ_2 is antisymmetric, i.e., $\gamma_1 = \gamma_1^t$, $\gamma_3 = \gamma_3^t$ and $\gamma_2 = -\gamma_2^t$. To construct the matrices for higher even dimensions, we then proceed by induction, setting:

$$\begin{aligned}\gamma_i^{(d+2)} = \sigma_1 \otimes \gamma_i^{(d)} &= \begin{pmatrix} 0 & \gamma_i^{(d)} \\ \gamma_i^{(d)} & 0 \end{pmatrix}, i = 1, 2, \dots, d+1, \\ \gamma_{d+2} = \sigma_2 \otimes I^{(d)} &= \begin{pmatrix} 0 & -iI_d \\ iI_d & 0 \end{pmatrix},\end{aligned}\tag{I-5}$$

where, I_d is the unit matrix in $2^{\frac{d}{2}}$ dimensions. As a consequence $\gamma_s^{(d+2)}$ has the form:

$$\gamma_s^{(d+2)} = \gamma_{d+3}^{(d+2)} = \sigma_3 \otimes I_d = \begin{pmatrix} I_d & 0 \\ 0 & -I_d \end{pmatrix}\tag{I-6}$$

A straightforward calculation shows that if the matrices $\gamma_i^{(d)}$ satisfy relations (I-3), the $\gamma_i^{(d+2)}$ matrices satisfy the same relations. By induction we see that the γ matrices are all hermitian. from (I-5), it is seen that, if $\gamma_i^{(d)}$ is symmetric or antisymmetric, $\gamma_i^{(d+2)}$ has the same property. The matrix $\gamma_{d+2}^{(d+2)}$ is antisymmetric and the matrix γ_{d+3}^{d+2} is symmetric. It follows immediately that, in this representation, all γ matrices with odd index are symmetric and all matrices with even index are antisymmetric, i.e.,

$$\gamma_i^t = (-1)^{i+1} \gamma_i.\tag{I-7}$$

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